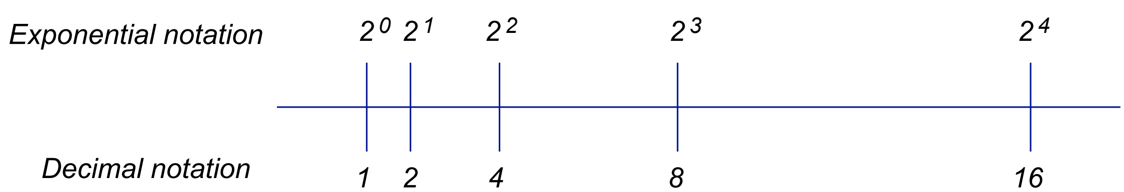
Using the concept of a number identifying a location on the number line (or a distance from 0).

<u>Pre-requisite knowledge</u>:

- Students should be familiar with exponential numbers and their real number equivalent for positive and negative exponents.
- Students should be familiar with exponential numbers with a positive base and negative exponents are all located between 0 and 1.
- Students should be familiar with the reciprocal relationship between positive and negative exponents.
- Students should be familiar with multiplying and dividing exponential numbers
- Students should know that a nth-root is a solution to a polynomial equation of degree n.
- Students should be familiar with locating exponential numbers on the number line.
- Students should be familiar with how exponential numbers are spaced on the number line as contrasted with integer and rational number spacing.
- Students should be familiar with geometric mean and arithmetical mean and how these are different.
- Students should be able to distinguish "in the middle" from "in between."

Figure 1



Notes with figure 1: While exponent 2 is in the middle of exponents 1 and 3, 2^2 is not in the middle of 2^1, and 2^3. Distances (differences) between consecutive exponential numbers grow by the rate of the base. This was the distance (difference) between a^{n+2} and a^{n+1} is *a* times larger than the distance (difference) between a^{n+1} and a^n.

To assess this understanding ask students the following:
Given the difference between 10^{60} and 10^{61} and the difference between 10^{90} and 10^{91}, which of the following is true:

A. $10^{61} - 10^{60} > 10^{91} - 10^{90}$
B. $10^{61} - 10^{60} = 10^{91} - 10^{90}$
C. $10^{61} - 10^{60} < 10^{91} - 10^{90}$

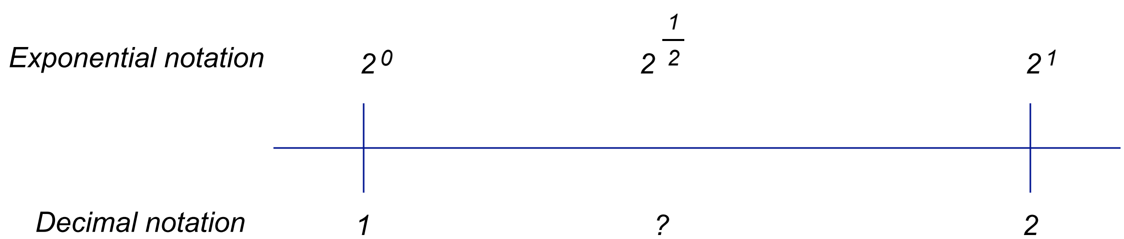
Students who do not understand the effect on spacing of exponential numbers will most often choose B, because they look at the difference of the exponents. You can help these students to think about how many whole numbers are between 10 and 100 (i.e. 90) and 100 and 1000 (900).

Introducing rational exponents conceptually

We are now going to focus on a section between two consecutive exponential numbers with integer exponents.
First ask students to identify numbers (and their locations) on the real number representation side (below) between 1 and 2.
Then ask students to identify a number between 2^0 and 2^1, using exponential notation. If necessary guide students to realize that that should be a rational exponent.
(Developmentally children should realize the number line to be continuous rather than discrete to participate in the line of reasoning. Also think of the completeness theorem from Archimedes.)

Then pose the question to the class to determine where $2^{\frac{1}{2}}$ is located between 2^0 and 2^1 and to find the decimal (real number) equivalent position. (See figure 2 below.)

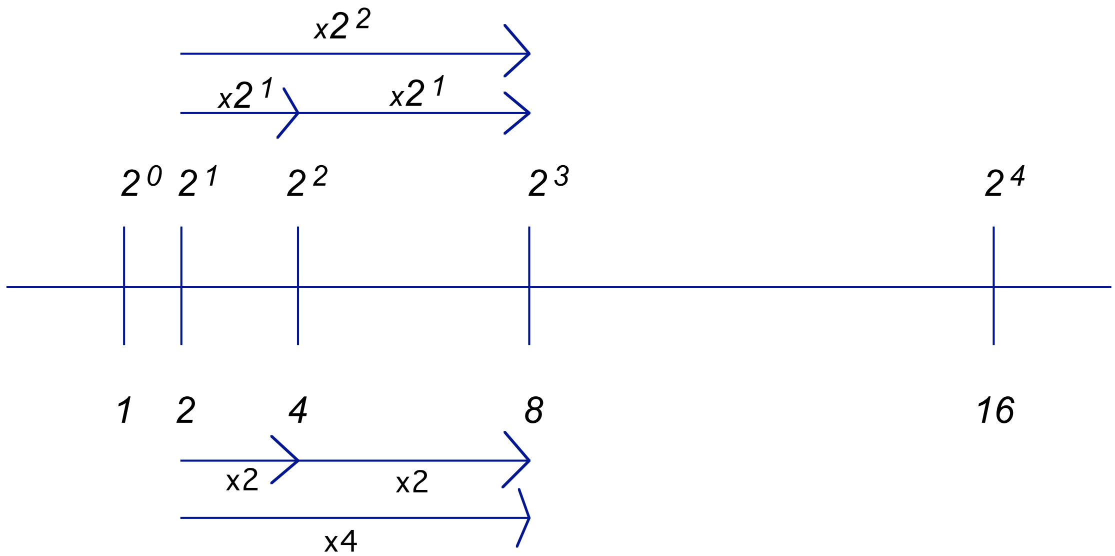
Figure 2



Many students will assume (as discussed above with integer exponents) that $2^{\frac{1}{2}}$ is in the middle of 2^0 and 2^1. So we need to scaffold toward the geometric mean using figure 1 and then apply this to figure 2 as follows:

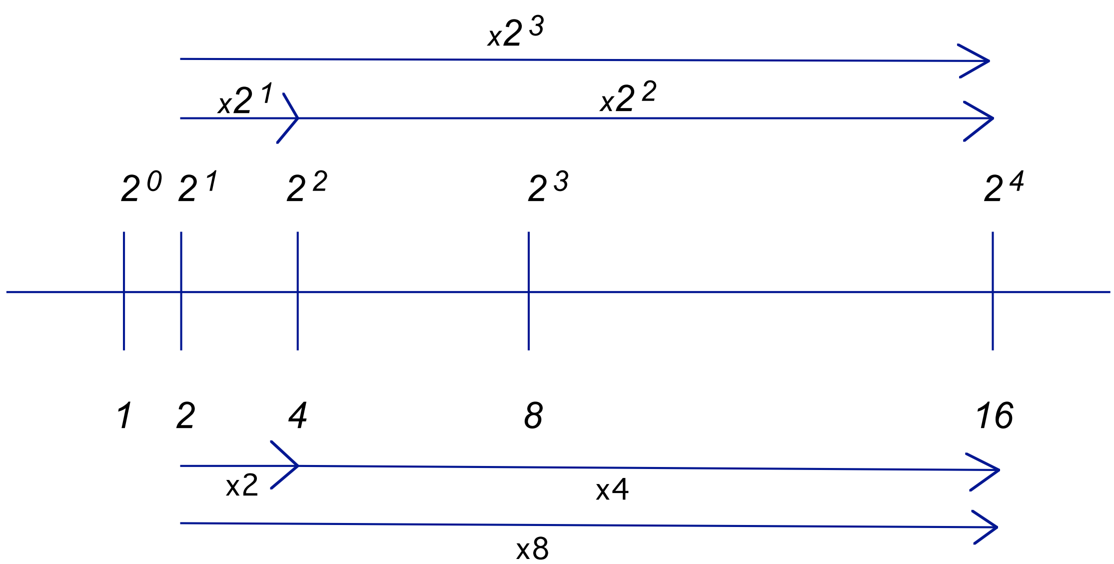
In figure 3 and figure 4 we can observe (determine) how the multipliers (rates or growth factors) relate at the real number side and the equivalent exponential side.
Here we need to point out the rule of how to multiply exponential numbers and how two subsequent multipliers combine to a single multiplier.

Figure 3



x2 2
x2 1
x2 1
2 0
2 1
2 2
2 3
2 4
1
2
4
8
16
x2
x2
x4

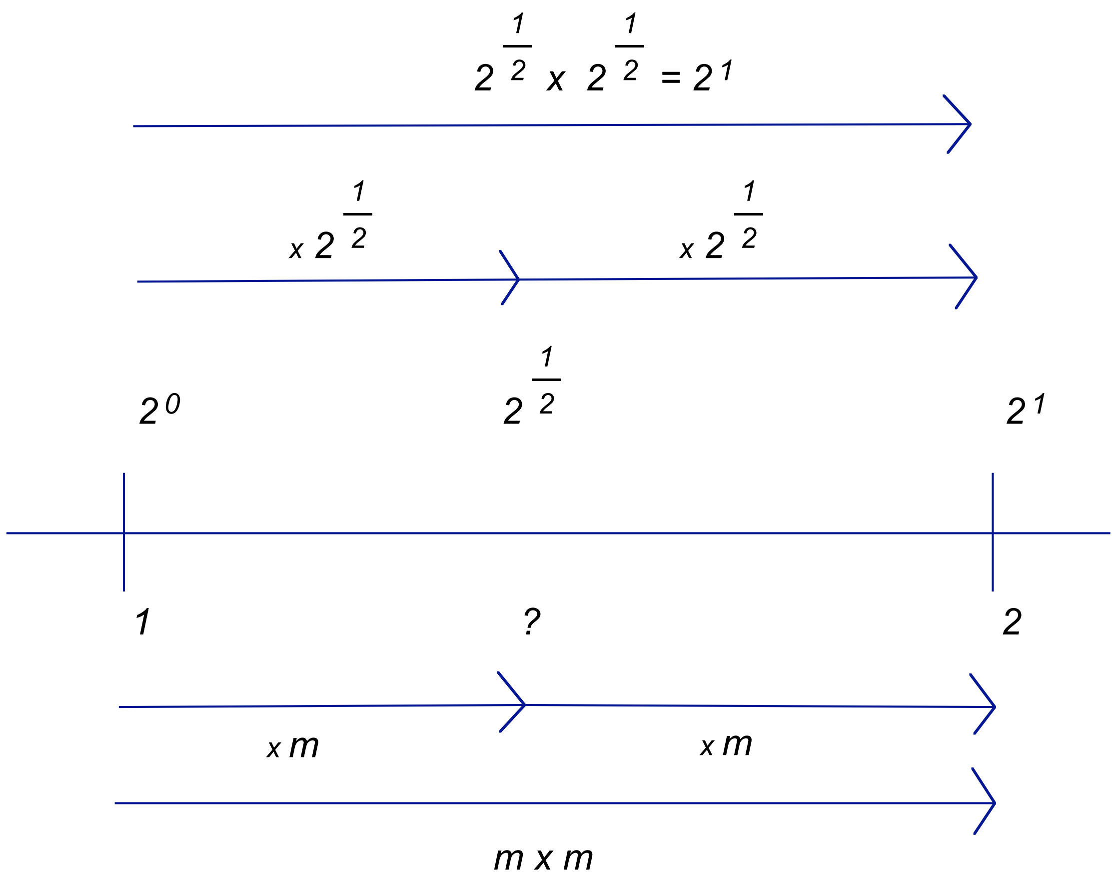
OR



x2 3
x2 1
x2 2
2 0
2 1
2 2
2 3
2 4
1
2
4
8
16
x2
x4
x8

In figure 4 we look at a specific portion of the number line and need to know that the rule for multiplying exponential numbers also works for rational exponents (and even real exponents later). We also need the idea of a geometric mean to determine the equivalent multiplier in the real number context.

Figure 4



From the figure above we can conclude that, if $m = 2^{\frac{1}{2}}$, then *m x m = 2*, which implies that $m^2 = 2$. From this we conclude that $m = \sqrt{2}$.
This then means that $2^{\frac{1}{2}} = \sqrt{2}$

Similarly we can determine that $2^{\frac{1}{3}} = \sqrt[3]{2}$ and so on.

All of the above can be derived in a real world context as well:

Today, in my pond I found 1 square foot of algae. A biologist told me that the amount of algae doubles each day this time of the year.

Possible questions:
1. What area of the pond will be covered with algae in *p* days from now? (whole number exponents)
2. What area of the pond was covered with algae *n* days ago? (negative integer exponents)
3. What area of the pond will be covered with algae1/2 day from now? (rational positive exponents)
4. What area of the pond was be covered with algae 1/2 days ago? (rational negative exponents)